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## $R$ Matrix for $U_q E_7$

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### Abstract

The quantum Clebsch-Gordan coefficients and the explicit form of the  $\check{R}_q$  matrix related with the minimal representation of the quantum enveloping algebra  $U_q E_7$  are calculated in this paper.

## 1. INTRODUCTION

Recently, quantum groups<sup>1</sup> have drawn an increasing attention by both physicists and mathematicians. A quantum group is introduced as the non-commutative and non-cocommutative Hopf algebra  $\mathcal{A}$  obtained by continuous deformations of the Hopf algebra of the function of a Lie group. The associative algebra  $\mathcal{A}$  is freely generated by non-commutating matrix entries  $T_b^a$ , satisfying the so-called  $RTT = TTR$  relation<sup>2</sup>:

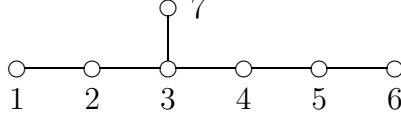
$$\left(\hat{R}_q\right)_{rs}^{ab} T_c^r T_d^s = T_r^a T_s^b \left(\hat{R}_q\right)_{cd}^{rs} \quad (1.1)$$

where  $\hat{R}_q$  matrix is a solution of the simple Yang-Baxter equation<sup>3</sup> related with the minimal representation of the corresponding quantum enveloping algebra.

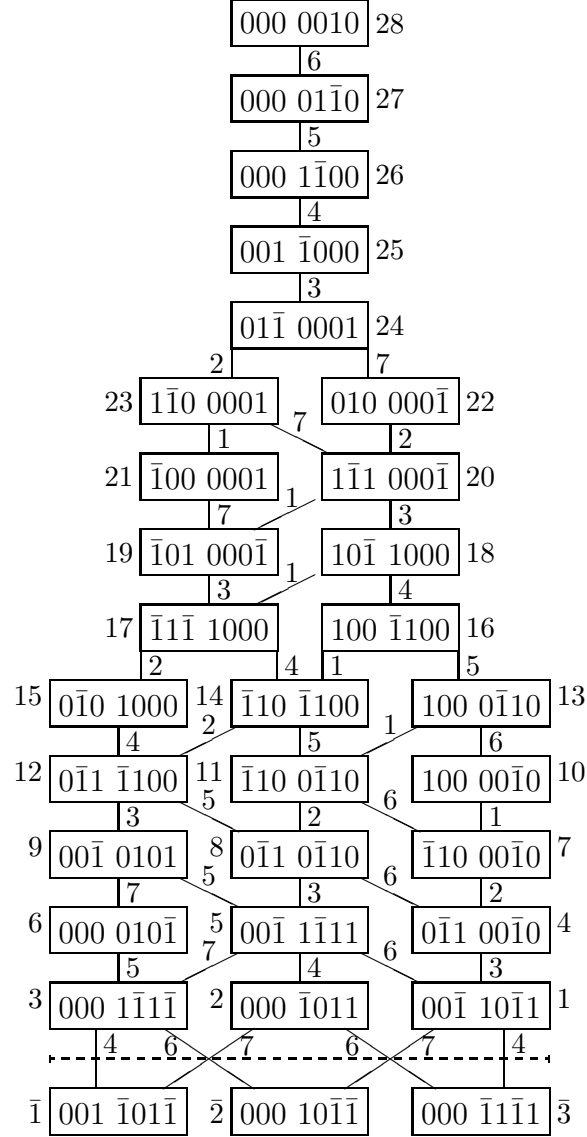
The  $\hat{R}_q$  matrices are well known for the quantum classical Lie enveloping algebras<sup>4</sup>. Based on those solutions and (1.1) the quantum groups  $A_\ell(q)$ ,  $B_\ell(q)$ ,  $C_\ell(q)$  and  $D_\ell(q)$ , and their bicovariant differential calculus were studied<sup>5,6</sup>. In order to generalize the concept of the quantum group into the quantum exceptional groups, one has to calculate the  $\hat{R}_q$  matrices for the quantum exceptional enveloping algebras firstly. Kuniba<sup>7</sup> calculated the quantum Clebsch-Gordan ( $q$ -CG) coefficients for the minimal representation of  $U_q G_2$ . Although he did not list the explicit elements of the  $\hat{R}_q$  matrix in his paper, it is not hard to calculate them from the  $q$ -CG coefficients. The explicit  $\hat{R}_q$  matrices for  $U_q F_4$  and  $U_q E_6$  were listed in Refs. [8] and [9], but partly<sup>9</sup> for  $U_q E_7$ . The most complicated  $56 \times 56$  submatrix did not given there. In the direct product space of two minimal representations spaces there are 56 states with weight zero. It is a very complicated task to combine them into orthogonal bases belonging to four irreducible representations, respectively. In the present paper we use a systematic method<sup>10</sup> to calculate both the representation matrix and the  $q$ -CG matrix simultaneously in terms of the Mathematica program, and succeed in obtaining the  $56 \times 56$  submatrix of the  $\hat{R}_q$  matrix.

The plan of this paper is as follows. In Sec. 2, we introduce our notation. The main results of  $q$ -CG coefficients and the  $\hat{R}_q$  matrix are listed in Sec.3 and Sec.4.

## 2. NOTATIONS



**Fig.1.1.** Dynkin diagram for  $E_7$ .



**Fig.2** Up-half block weight diagram for the highest weight representation  $\lambda_6$  of  $U_q E_7$

The Dynkin diagram for the exceptional group  $E_7$  is showed in Fig.1. There are seven simple roots  $\mathbf{r}_j$  and seven fundamental dominant weights  $\lambda_j$  in  $E_7$  that are related by the Cartan matrix  $A_{ij}$ :

$$\mathbf{r}_j = \sum_{i=1}^7 \lambda_i A_{ij} \quad (2.1)$$

An irreducible representation is denoted by the highest weight  $\mathbf{M}$  that is an non-negative integral combination of the fundamental weights  $\lambda_j$ , and the states in the representation by weights  $\mathbf{m}$  that are integral combinations of  $\lambda_j$ . The minimal representation of  $E_7$  is  $\lambda_6$  that is 56 dimensional. Since the block weight diagram of the minimal representation is symmetry up and down, only up-half part of the diagram is given in Fig.2.

To make notation simpler we enumerate the states by one index  $a$ :

$$a = 28, 27, \dots, 2, 1, \bar{1}, \bar{2}, \dots, \bar{28}, \quad \bar{a} \equiv -a \quad (2.2)$$

The index  $a$  is written by the blocks in Fig.2. Denote by  $\mathbf{m}_a$  the weight of the state  $a$ , and by  $h_a$  the height of the state. The weights of the states are filled in the blocks of Fig.2. For example,  $\mathbf{m}_{28} = \lambda_6$ , and  $\mathbf{m}_{23} = \lambda_1 - \lambda_2 + \lambda_7$ . The heights of the states are calculated by:

$$\mathbf{M} - \mathbf{m}_a = \sum_{j=1}^7 c_j \mathbf{r}_j, \quad h_a = \sum_{j=1}^7 c_j$$

and listed as follows:

$$\begin{aligned} h_{28} &= 0, & h_{27} &= 1, & h_{26} &= 2, \\ h_{25} &= 3, & h_{24} &= 4, & h_{23} &= h_{22} = 5, \\ h_{21} &= h_{20} = 6, & h_{19} &= h_{18} = 7, & h_{17} &= h_{16} = 8, \\ h_{15} &= h_{14} = h_{13} = 9, & h_{12} &= h_{11} = h_{10} = 10, & h_9 &= h_8 = h_7 = 11, \\ h_6 &= h_5 = h_4 = 12, & h_3 &= h_2 = h_1 = 13, & h_{\bar{a}} &= 27 - h_a, \end{aligned} \quad (2.3)$$

In Fig.2, some neighboured blocks are connected by a line. As well known<sup>10</sup>, in the quantum enveloping algebra  $U_q E_7$  there are generators  $e_j$ ,  $f_j$  and  $k_j$ ,  $1 \leq j \leq 7$ , satisfying standard quantum algebraic relations<sup>10</sup>. The line connecting two blocks in Fig.2, labeled a number  $j$ , denotes that two states in the blocks can be related by a raising (or lowering) operator  $e_j$  (or  $f_j$ ). For example, the state 20 acted by  $e_2$  becomes the state 22. Now, the lines in Fig.2 denotes the condition that the quantum representation matrix elements of  $e_j$  and  $f_j$  related with those two states in the minimal representation is non-vanishing, namely,

$$D_q(e_j)_{a \ b} = D_q(f_j)_{b \ a} = 1 \quad (2.4)$$

if  $a > b$  and two blocks filled by  $\mathbf{m}_a$  and  $\mathbf{m}_b$  are connected by a line labeled by  $j$ . Otherwise the matrix elements are vanishing. The quantum representation matrices of  $k_j$  are diagonal, and the diagonal elements depend on the weights:

$$D_q(k_j)_{a \ a} = q^{n_j/2}, \quad \text{if } \mathbf{m}_a = \sum_{j=1}^7 n_j \lambda_j \quad (2.5)$$

The direct product of two minimal representations is a reducible representation with the Clebsch-Gordan series as follows:

$$\lambda_6 \otimes \lambda_6 = (2\lambda_6) \oplus \lambda_5 \oplus \lambda_1 \oplus \mathbf{0} \quad (2.6)$$

The dimensions of the representations  $(2\lambda_6)$ ,  $\lambda_5$ ,  $\lambda_1$ , and  $\mathbf{0}$  are 1463, 1539, 133 and 1, respectively. Their Casimir are 30, 28, 18, and 1, respectively.

The solution of the simple Yang-Baxter equation related to the minimal representation of  $U_q E_7$  can be calculated by a standard method<sup>11,10</sup>:

$$\begin{aligned} \check{R}_q &= \mathcal{P}_{2\lambda_6} - q^2 \mathcal{P}_{\lambda_5} + q^{12} \mathcal{P}_{\lambda_1} - q^{30} \mathcal{P}_{\mathbf{0}} \\ (\mathcal{P}_{\mathbf{N}})_{cd}^{ab} &= (C_q)_{\mathbf{m}_a \mathbf{m}_b \mathbf{N}(\mathbf{m}_a + \mathbf{m}_b)} (C_q)_{\mathbf{m}_c \mathbf{m}_d \mathbf{N}(\mathbf{m}_c + \mathbf{m}_d)} \end{aligned} \quad (2.7)$$

where  $\mathcal{P}_{\mathbf{N}}$  is a projection operator, and  $(C_q)_{\mathbf{N}}$  is the  $q$ -CG matrix reduces the product representation  $\lambda_6 \otimes \lambda_6$  into the irreducible one  $\mathbf{N}$ :

$$| \mathbf{N}, \mathbf{m} \rangle = \sum_{\mathbf{m}_a} | \lambda_6, \mathbf{m}_a \rangle | \lambda_6, (\mathbf{m} - \mathbf{m}_a) \rangle (C_q)_{\mathbf{m}_a (\mathbf{m} - \mathbf{m}_a) \mathbf{N} \mathbf{m}} \quad (2.8)$$

For convenience, we usually denote by their indices  $a$  the states on the right hand side of (2.8), for example see (3.1).

Our main problems are to calculate the  $q$ -CG coefficients by (2.8), and then, to calculate the  $\check{R}_q$  matrix by (2.7).

### 3. QUANTUM CLEBSCH-GORDAN COEFFICIENTS

The state with the highest weight in the product space is single:

$$| 2\lambda_6, 2\lambda_6 \rangle = | 28 \rangle | 28 \rangle \quad (3.1)$$

Acting the lowering operator  $f_j$  on (3.1), we are able to calculate the expansions for other states. Since the Weyl equivalent states have the same expansions, we only need to list the expansions for the states with the dominant weights.

The second dominant weight is  $\lambda_5$ . Only two representations in the CG series (2.6) have this dominant weight:

$$\begin{aligned} |2\lambda_6, \lambda_5\rangle &= [2]^{-1/2} \{q^{1/2} |28\rangle |27\rangle + q^{-1/2} |27\rangle |28\rangle\} \\ |\lambda_5, \lambda_5\rangle &= [2]^{-1/2} \{q^{-1/2} |28\rangle |27\rangle - q^{1/2} |27\rangle |28\rangle\} \end{aligned} \quad (3.2)$$

where and hereafter we use the following notation as usual:

$$\omega = q - q^{-1}, \quad [n] = \omega^{-1} (q^n - q^{-n}) \quad (3.3)$$

The second expansion in (3.2) is calculated from the orthogonality of  $q$ -CG coefficients. Note that the second half terms in the expansions (3.2) can be obtained from the first half terms by changing the order of two states and replacing  $q$  by  $q^{-1}$ . The representation  $(2\lambda_6)$  is symmetric so that the terms in the second half have the same sign as their partner, and the representation  $\lambda_5$  is antisymmetric so that the partners have opposite signs. In the following expansions we will replace the second half terms by (*sym. terms*) or (*antisym. terms*) for simplicity.

Multiplicity appears in the states with the third dominant weight  $\lambda_1$ . We distinguish the multiple states by a subscript  $j$  if the state is obtained by the lowering operator  $f_j$  from a state with a height lower by one. The expansions of those states are as follows:

$$\begin{aligned} |2\lambda_6, \lambda_1\rangle_3 &= [2]^{-1} \{q |25\rangle |18\rangle + |24\rangle |20\rangle + (\text{sym. terms})\} \\ |2\lambda_6, \lambda_1\rangle_2 &= [2]^{-1}[3]^{-1/2} \{-q |25\rangle |18\rangle + q^2 |24\rangle |20\rangle \\ &\quad + [2] |23\rangle |22\rangle + (\text{sym. terms})\} \\ |2\lambda_6, \lambda_1\rangle_4 &= ([4][3][2])^{-1/2} \{q[3] |26\rangle |16\rangle + q^{-2} |25\rangle |18\rangle \\ &\quad - q^{-1} |24\rangle |20\rangle + |23\rangle |22\rangle + (\text{sym. terms})\} \\ |2\lambda_6, \lambda_1\rangle_5 &= ([5][4][2])^{-1/2} \{q[4] |27\rangle |13\rangle + q^{-3} |26\rangle |16\rangle \\ &\quad - q^{-2} |25\rangle |18\rangle + q^{-1} |24\rangle |20\rangle - |23\rangle |22\rangle + (\text{sym. terms})\} \\ |2\lambda_6, \lambda_1\rangle_6 &= ([6][5][2])^{-1/2} \{q[5] |28\rangle |10\rangle + q^{-4} |27\rangle |13\rangle \\ &\quad - q^{-3} |26\rangle |16\rangle + q^{-2} |25\rangle |18\rangle - q^{-1} |24\rangle |20\rangle \\ &\quad + |23\rangle |22\rangle + (\text{sym. terms})\} \end{aligned} \quad (3.4)$$

$$\begin{aligned}
|\lambda_5, \lambda_1\rangle_3 &= [2]^{-1} \{ |25\rangle |18\rangle + q^{-1} |24\rangle |20\rangle + (\text{antisym. terms}) \} \\
|\lambda_5, \lambda_1\rangle_2 &= [2]^{-1} [3]^{-1/2} \{ - |25\rangle |18\rangle + q |24\rangle |20\rangle \\
&\quad + q^{-1} [2] |23\rangle |22\rangle + (\text{antisym. terms}) \} \\
|\lambda_5, \lambda_1\rangle_7 &= \left( \frac{[2]}{[4][3]} \right)^{1/2} \{ - |25\rangle |18\rangle + q |24\rangle |20\rangle \\
&\quad - q^2 |23\rangle |22\rangle + (\text{antisym. terms}) \} \\
|\lambda_5, \lambda_1\rangle_4 &= \left( \frac{[2]}{[6][4]} \right)^{1/2} \{ ([4]/[2]) |26\rangle |16\rangle - q^{-2} \omega |25\rangle |18\rangle \\
&\quad + q^{-1} \omega |24\rangle |20\rangle - \omega |23\rangle |22\rangle + (\text{antisym. terms}) \} \\
|\lambda_5, \lambda_1\rangle_5 &= \left( \frac{[4][3]}{[8][6][2]} \right)^{1/2} \{ ([6]/[3]) |27\rangle |13\rangle - q^{-3} \omega |26\rangle |16\rangle \\
&\quad + q^{-2} \omega |25\rangle |18\rangle - q^{-1} \omega |24\rangle |20\rangle \\
&\quad + \omega |23\rangle |22\rangle + (\text{antisym. terms}) \} \\
|\lambda_5, \lambda_1\rangle_6 &= \left( \frac{[5][4]}{[10][8][2]} \right)^{-1/2} \{ ([8]/[4]) |28\rangle |10\rangle - q^{-4} \omega |27\rangle |13\rangle \\
&\quad + q^{-3} \omega |26\rangle |16\rangle - q^{-2} \omega |25\rangle |18\rangle + q^{-1} \omega |24\rangle |20\rangle \\
&\quad - \omega |23\rangle |22\rangle + (\text{antisym. terms}) \}
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
|\lambda_1, \lambda_1\rangle_7 &= \left( \frac{[5]}{[10][6]} \right)^{1/2} \{ q^{-5} |28\rangle |10\rangle - q^{-4} |27\rangle |13\rangle \\
&\quad + q^{-3} |26\rangle |16\rangle - q^{-2} |25\rangle |18\rangle + q^{-1} |24\rangle |20\rangle \\
&\quad - |23\rangle |22\rangle + (\text{sym. terms}) \}
\end{aligned} \tag{3.6}$$

The fourth dominant weight is  $\mathbf{0}$ . There are 56 states with this weight. Here we only list an expansion for a state of the one-dimensional representation  $\mathbf{0}$ :

$$\begin{aligned}
|\mathbf{0}, \mathbf{0}\rangle_7 &= \left( \frac{[9][5]}{[18][14][10]} \right)^{1/2} \{ q^{-27/2} |28\rangle |\bar{28}\rangle - q^{-25/2} |27\rangle |\bar{27}\rangle + q^{-23/2} |26\rangle |\bar{26}\rangle \\
&\quad - q^{-21/2} |25\rangle |\bar{25}\rangle + q^{-19/2} |24\rangle |\bar{24}\rangle - q^{-17/2} |23\rangle |\bar{23}\rangle - q^{-17/2} |22\rangle |\bar{22}\rangle \\
&\quad + q^{-15/2} |21\rangle |\bar{21}\rangle + q^{-15/2} |20\rangle |\bar{20}\rangle - q^{-13/2} |19\rangle |\bar{19}\rangle - q^{-13/2} |18\rangle |\bar{18}\rangle \\
&\quad + q^{-11/2} |17\rangle |\bar{17}\rangle + q^{-11/2} |16\rangle |\bar{16}\rangle - q^{-9/2} |15\rangle |\bar{15}\rangle - q^{-9/2} |14\rangle |\bar{14}\rangle \\
&\quad - q^{-9/2} |13\rangle |\bar{13}\rangle + q^{-7/2} |12\rangle |\bar{12}\rangle + q^{-7/2} |11\rangle |\bar{11}\rangle + q^{-7/2} |10\rangle |\bar{10}\rangle \\
&\quad - q^{-5/2} |9\rangle |\bar{9}\rangle - q^{-5/2} |8\rangle |\bar{8}\rangle - q^{-5/2} |7\rangle |\bar{7}\rangle + q^{-3/2} |6\rangle |\bar{6}\rangle \\
&\quad + q^{-3/2} |5\rangle |\bar{5}\rangle + q^{-3/2} |4\rangle |\bar{4}\rangle - q^{-1/2} |3\rangle |\bar{3}\rangle - q^{-1/2} |2\rangle |\bar{2}\rangle \\
&\quad - q^{-1/2} |1\rangle |\bar{1}\rangle + (\text{antisym. terms}) \}
\end{aligned} \tag{3.7}$$

#### 4. SOLUTION OF THE YANG-BAXTER EQUATION

Now, we are able to calculate the solution  $\check{R}_q$  of the simple Yang-Baxter equation

from (2.7). Firstly, from (2.7)  $\check{R}_q$  is a symmetric matrix:

$$\left(\check{R}_q\right)_{bd}^{ac} = \left(\check{R}_q\right)_{ac}^{bd} \quad (4.1)$$

Secondly, from the general properties <sup>2</sup> of  $\check{R}_q$ , we know:

$$\left(\check{R}_q\right)_{bd}^{ac} = 0, \quad \text{if } c < b \quad (4.2)$$

At last, the  $q$ -CG coefficients given in Sec.3 are invariant in the Weyl reflection, so do  $\check{R}_q$ . We only need to calculate the  $\check{R}_q$  matrix elements for the dominant weights. According to the Weyl orbit sizes of the dominant weights, we know that the  $\check{R}_q$  matrix is a block matrix with 56 1-dimensional submatrices, 756 2-dimensional submatrices, 126 12-dimensional submatrices, and one 56-dimensional submatrix.

i)  $1 \times 1$  submatrix for the dominant weight  $(2\lambda_6)$

From (3.1) we have:

$$\left(\check{R}_q\right)_{28\ 28}^{28\ 28} = 1 \quad (4.3)$$

From the Weyl reflection, (4.3) holds if one replaces 28 by  $a$ , where  $a$  runs over 28 to  $\bar{28}$ .

ii)  $2 \times 2$  submatrix for the dominant weight  $\lambda_5$

From (3.2) we have:

$$\left(\check{R}_q\right)_{27\ 28}^{28\ 27} = 1, \quad \left(\check{R}_q\right)_{27\ 28}^{27\ 28} = -q \omega = 1 - q^2 \quad (4.4)$$

iii)  $12 \times 12$  submatrix for the dominant weight  $\lambda_1$

As shown in (3.4) to (3.6), in the direct product space there are 12 states with the dominant weight  $\lambda_1$ . They are:

$$\begin{aligned} (a, a') = & (28, 10), (27, 12), (26, 16), (25, 18), (24, 20), (23, 22), \\ & (22, 23), (20, 24), (18, 25), (16, 26), (12, 27), (10, 28) \end{aligned} \quad (4.5)$$

The rows (columns) of the submatrix with this dominant weight are also denoted by those pair of numbers  $(a, a')$ . We would like to emphasize that the number  $a'$  is unique determined by the number  $a$ . Now, the calculation results for the submatrix of  $\check{R}_q$  are



as follows:

$$\left(\check{R}_q\right)_{bb'}^{aa'} = \begin{cases} q^2, & \text{if } h_a + h_b = 10 \text{ and } a \neq b \\ -(-q)^{h_a+h_b-9}\omega, & \text{if } h_a + h_b > 10 \text{ and } a \neq b \\ q^{h_a-4}\omega^2[h_a - 5], & \text{if } a = b \text{ and } h_a \geq 6 \\ 0, & \text{the rest cases} \end{cases} \quad (4.6)$$

where the heights  $h_a$  of the states  $a$  are given in (2.3).

iv)  $56 \times 56$  submatrix for the dominant weight  $\mathbf{0}$

The rows (columns) of this submatrix are denoted by  $(a, \bar{a})$ . The calculation results for the  $56 \times 56$  submatrix of  $\check{R}_q$  are as follows. There are five cases for the submatrix elements.

a) If  $\mathbf{m}_a + \mathbf{m}_b$  is not a non-positive combination of the simple roots  $\mathbf{r}_j$  of  $E_7$ , i.e., there is at least one positive coefficient in the combination, we have:

$$\left(\check{R}_q\right)_{b\bar{b}}^{a\bar{a}} = 0 \quad (4.7a)$$

b) If  $a = -b$ , we have:

$$\left(\check{R}_q\right)_{\bar{a}a}^{a\bar{a}} = q^3 \quad (4.7b)$$

c) If  $a = b$  and  $\mathbf{m}_a + \mathbf{m}_b$  is a non-positive combination of the simple roots  $\mathbf{r}_j$  of  $E_7$ ,

we have:

$$\begin{aligned} \left(\check{R}_q\right)_{\bar{55}}^{55} &= -q^3\omega^3, & \left(\check{R}_q\right)_{\bar{88}}^{88} &= -q^4[2]\omega^3, & \left(\check{R}_q\right)_{\bar{99}}^{99} &= -q^4[2]\omega^3, \\ \left(\check{R}_q\right)_{\bar{1111}}^{1111} &= -q^5[3]\omega^3, & \left(\check{R}_q\right)_{\bar{1212}}^{1212} &= -q^5[2]^2\omega^3, & \left(\check{R}_q\right)_{\bar{1313}}^{1313} &= -q^6[4]\omega^3, \\ \left(\check{R}_q\right)_{\bar{1414}}^{1414} &= -q^6[3][2]\omega^3, & \left(\check{R}_q\right)_{\bar{1515}}^{1515} &= -q^6[3][2]\omega^3, & \left(\check{R}_q\right)_{\bar{1616}}^{1616} &= -q^7[4][2]\omega^3, \\ \left(\check{R}_q\right)_{\bar{1717}}^{1717} &= -q^7[3]^2\omega^3, & \left(\check{R}_q\right)_{\bar{1818}}^{1818} &= -q^8[4][3]\omega^3, & \left(\check{R}_q\right)_{\bar{1919}}^{1919} &= -q^8[4][3]\omega^3, \\ \left(\check{R}_q\right)_{\bar{2020}}^{2020} &= -q^9[4]^2\omega^3, & \left(\check{R}_q\right)_{\bar{2121}}^{2121} &= -q^9[5][3]\omega^3, & \left(\check{R}_q\right)_{\bar{2222}}^{2222} &= -q^{10}[5][4]\omega^3, \\ \left(\check{R}_q\right)_{\bar{2323}}^{2323} &= -q^{10}[5][4]\omega^3, & \left(\check{R}_q\right)_{\bar{2424}}^{2424} &= -q^{11}[5]^2\omega^3, & \left(\check{R}_q\right)_{\bar{2525}}^{2525} &= -q^{12}[6][5]\omega^3, \\ \left(\check{R}_q\right)_{\bar{2626}}^{2626} &= -q^{13}[7][5]\omega^3, & \left(\check{R}_q\right)_{\bar{2727}}^{2727} &= -q^{14}[8][5]\omega^3, & \left(\check{R}_q\right)_{\bar{2828}}^{2828} &= -q^{15}[9][5]\omega^3, \end{aligned} \quad (4.7c)$$

d) If  $\mathbf{m}_a + \mathbf{m}_b$  is a negative root of  $E_7$ , we have:

$$\left(\check{R}_q\right)_{b\bar{b}}^{a\bar{a}} = (-q)^{h_a+h_b-25}\omega \quad (4.7d)$$

e) If  $a \neq \pm b$ , and  $\mathbf{m}_a + \mathbf{m}_b$  is a non-positive combination of the simple roots  $\mathbf{r}_j$ , but not a negative root of  $E_7$ , we have:

$$\left(\check{R}_q\right)_{b\bar{b}}^{a\bar{a}} = (-q)^{h_a+h_b-25}q^{-\alpha}[\alpha]\omega^2 \quad (4.7e)$$

where the following pairs  $(a, b)$  correspond to  $\alpha = 1$ :

$$\begin{aligned}
&(\bar{5}, \bar{1}), (\bar{5}, \bar{2}), (\bar{5}, \bar{3}), (\bar{8}, \bar{1}), (\bar{8}, \bar{2}), (\bar{8}, \bar{3}), (\bar{8}, \bar{5}), (\bar{9}, \bar{3}), \\
&(\bar{9}, \bar{1}), (\bar{9}, \bar{2}), (\bar{9}, \bar{5}), (\bar{11}, \bar{4}), (\bar{11}, \bar{1}), (\bar{11}, \bar{2}), (\bar{11}, \bar{3}), (\bar{11}, \bar{5}), \\
&(\bar{11}, \bar{8}), (\bar{12}, \bar{5}), (\bar{12}, \bar{3}), (\bar{12}, \bar{1}), (\bar{12}, \bar{2}), (\bar{13}, \bar{7}), (\bar{13}, \bar{4}), (\bar{13}, \bar{1}), \\
&(\bar{13}, \bar{2}), (\bar{13}, \bar{3}), (\bar{13}, \bar{5}), (\bar{13}, \bar{8}), (\bar{13}, \bar{11}), (\bar{14}, \bar{8}), (\bar{14}, \bar{5}), (\bar{14}, \bar{4}), \\
&(\bar{14}, \bar{3}), (\bar{14}, \bar{1}), (\bar{14}, \bar{2}), (\bar{16}, \bar{11}), (\bar{16}, \bar{8}), (\bar{16}, \bar{7}), (\bar{16}, \bar{5}), (\bar{16}, \bar{4}), \\
&(\bar{16}, \bar{3}), (\bar{16}, \bar{1}), (\bar{16}, \bar{2}), (\bar{17}, \bar{12}), (\bar{17}, \bar{8}), (\bar{17}, \bar{4}), (\bar{18}, \bar{14}), (\bar{18}, \bar{12}), \\
&(\bar{18}, \bar{11}), (\bar{18}, \bar{8}), (\bar{18}, \bar{7}), (\bar{18}, \bar{4}), (\bar{20}, \bar{17}), (\bar{20}, \bar{14}), (\bar{20}, \bar{11}), (\bar{20}, \bar{7}), \\
&(\bar{23}, \bar{19}), (\bar{23}, \bar{17}), (\bar{23}, \bar{14}), (\bar{23}, \bar{11}), (\bar{23}, \bar{7}), (\bar{24}, \bar{20}), (\bar{24}, \bar{19}),
\end{aligned}$$

the following pairs  $(a, b)$  correspond to  $\alpha = 2$ :

$$\begin{aligned}
&(\bar{8}, \bar{4}), (\bar{9}, \bar{6}), (\bar{12}, \bar{4}), (\bar{12}, \bar{6}), (\bar{12}, \bar{8}), (\bar{12}, \bar{9}), (\bar{14}, \bar{6}), (\bar{14}, \bar{9}), \\
&(\bar{14}, \bar{12}), (\bar{15}, \bar{2}), (\bar{15}, \bar{1}), (\bar{15}, \bar{3}), (\bar{15}, \bar{4}), (\bar{15}, \bar{5}), (\bar{15}, \bar{6}), (\bar{15}, \bar{8}), \\
&(\bar{15}, \bar{9}), (\bar{15}, \bar{12}), (\bar{16}, \bar{6}), (\bar{16}, \bar{9}), (\bar{16}, \bar{12}), (\bar{16}, \bar{14}), (\bar{17}, \bar{2}), (\bar{17}, \bar{1}), \\
&(\bar{17}, \bar{3}), (\bar{17}, \bar{5}), (\bar{17}, \bar{6}), (\bar{17}, \bar{9}), (\bar{18}, \bar{2}), (\bar{18}, \bar{1}), (\bar{18}, \bar{3}), (\bar{18}, \bar{5}), \\
&(\bar{18}, \bar{6}), (\bar{18}, \bar{9}), (\bar{19}, \bar{9}), (\bar{19}, \bar{5}), (\bar{19}, \bar{2}), (\bar{19}, \bar{1}), (\bar{19}, \bar{3}), (\bar{19}, \bar{6}), \\
&(\bar{20}, \bar{9}), (\bar{20}, \bar{5}), (\bar{20}, \bar{2}), (\bar{20}, \bar{1}), (\bar{20}, \bar{3}), (\bar{20}, \bar{6}), (\bar{22}, \bar{15}), (\bar{22}, \bar{12}), \\
&(\bar{22}, \bar{9}), (\bar{22}, \bar{8}), (\bar{22}, \bar{5}), (\bar{22}, \bar{4}), (\bar{22}, \bar{2}), (\bar{22}, \bar{1}), (\bar{22}, \bar{3}), (\bar{22}, \bar{6}), \\
&(\bar{24}, \bar{15}), (\bar{24}, \bar{12}), (\bar{24}, \bar{8}), (\bar{24}, \bar{4}), (\bar{25}, \bar{18}), (\bar{25}, \bar{17}), (\bar{25}, \bar{15})
\end{aligned}$$

the following pairs  $(a, b)$  correspond to  $\alpha = 3$ :

$$\begin{aligned}
&(\bar{11}, \bar{7}), (\bar{14}, \bar{7}), (\bar{14}, \bar{11}), (\bar{17}, \bar{7}), (\bar{17}, \bar{11}), (\bar{17}, \bar{14}), (\bar{17}, \bar{15}), (\bar{18}, \bar{15}), \\
&(\bar{18}, \bar{17}), (\bar{19}, \bar{4}), (\bar{19}, \bar{7}), (\bar{19}, \bar{8}), (\bar{19}, \bar{11}), (\bar{19}, \bar{12}), (\bar{19}, \bar{14}), (\bar{19}, \bar{15}), \\
&(\bar{19}, \bar{17}), (\bar{20}, \bar{4}), (\bar{20}, \bar{8}), (\bar{20}, \bar{12}), (\bar{20}, \bar{15}), (\bar{21}, \bar{6}), (\bar{21}, \bar{3}), (\bar{21}, \bar{1}), \\
&(\bar{21}, \bar{2}), (\bar{21}, \bar{4}), (\bar{21}, \bar{5}), (\bar{21}, \bar{7}), (\bar{21}, \bar{8}), (\bar{21}, \bar{9}), (\bar{21}, \bar{11}), (\bar{21}, \bar{12}), \\
&(\bar{21}, \bar{14}), (\bar{21}, \bar{15}), (\bar{21}, \bar{17}), (\bar{21}, \bar{19}), (\bar{23}, \bar{6}), (\bar{23}, \bar{3}), (\bar{23}, \bar{1}), (\bar{23}, \bar{2}), \\
&(\bar{23}, \bar{4}), (\bar{23}, \bar{5}), (\bar{23}, \bar{8}), (\bar{23}, \bar{9}), (\bar{23}, \bar{12}), (\bar{23}, \bar{15}), (\bar{24}, \bar{6}), (\bar{24}, \bar{3}), \\
&(\bar{24}, \bar{1}), (\bar{24}, \bar{2}), (\bar{24}, \bar{5}), (\bar{24}, \bar{9}), (\bar{25}, \bar{9}), (\bar{25}, \bar{6}), (\bar{25}, \bar{5}), (\bar{25}, \bar{3}), \\
&(\bar{25}, \bar{1}), (\bar{25}, \bar{2}), (\bar{26}, \bar{16}), (\bar{26}, \bar{14}), (\bar{26}, \bar{12}), (\bar{26}, \bar{9}), (\bar{26}, \bar{6}),
\end{aligned}$$

the following pairs  $(a, b)$  correspond to  $\alpha = 4$ :

$$\begin{aligned}
& (\bar{13}, \bar{10}), (\bar{16}, \bar{10}), (\bar{16}, \bar{13}), (\bar{18}, \bar{10}), (\bar{18}, \bar{13}), (\bar{18}, \bar{16}), (\bar{20}, \bar{10}), (\bar{20}, \bar{13}), \\
& (\bar{20}, \bar{16}), (\bar{20}, \bar{18}), (\bar{20}, \bar{19}), (\bar{22}, \bar{7}), (\bar{22}, \bar{10}), (\bar{22}, \bar{11}), (\bar{22}, \bar{13}), (\bar{22}, \bar{14}), \\
& (\bar{22}, \bar{16}), (\bar{22}, \bar{17}), (\bar{22}, \bar{18}), (\bar{22}, \bar{19}), (\bar{22}, \bar{20}), (\bar{23}, \bar{10}), (\bar{23}, \bar{13}), (\bar{23}, \bar{16}), \\
& (\bar{23}, \bar{18}), (\bar{23}, \bar{20}), (\bar{24}, \bar{7}), (\bar{24}, \bar{10}), (\bar{24}, \bar{11}), (\bar{24}, \bar{13}), (\bar{24}, \bar{14}), (\bar{24}, \bar{16}), \\
& (\bar{24}, \bar{17}), (\bar{24}, \bar{18}), (\bar{25}, \bar{4}), (\bar{25}, \bar{7}), (\bar{25}, \bar{8}), (\bar{25}, \bar{10}), (\bar{25}, \bar{11}), (\bar{25}, \bar{12}), \\
& (\bar{25}, \bar{13}), (\bar{25}, \bar{14}), (\bar{25}, \bar{16}), (\bar{26}, \bar{2}), (\bar{26}, \bar{1}), (\bar{26}, \bar{3}), (\bar{26}, \bar{4}), (\bar{26}, \bar{5}), \\
& (\bar{26}, \bar{7}), (\bar{26}, \bar{8}), (\bar{26}, \bar{10}), (\bar{26}, \bar{11}), (\bar{26}, \bar{13}), (\bar{27}, \bar{13}), (\bar{27}, \bar{11}), (\bar{27}, \bar{8}), \\
& (\bar{27}, \bar{5}), (\bar{27}, \bar{3}), (\bar{27}, \bar{2}), (\bar{27}, \bar{1}), (\bar{27}, \bar{4}), (\bar{27}, \bar{7}), (\bar{27}, \bar{10}),
\end{aligned}$$

and at last, the following pairs  $(a, b)$  correspond to  $\alpha = 5$ :

$$\begin{aligned}
& (\bar{23}, \bar{21}), (\bar{24}, \bar{21}), (\bar{24}, \bar{22}), (\bar{24}, \bar{23}), (\bar{25}, \bar{19}), (\bar{25}, \bar{20}), (\bar{25}, \bar{21}), (\bar{25}, \bar{22}), \\
& (\bar{25}, \bar{23}), (\bar{25}, \bar{24}), (\bar{26}, \bar{15}), (\bar{26}, \bar{17}), (\bar{26}, \bar{18}), (\bar{26}, \bar{19}), (\bar{26}, \bar{20}), (\bar{26}, \bar{21}), \\
& (\bar{26}, \bar{22}), (\bar{26}, \bar{23}), (\bar{26}, \bar{24}), (\bar{26}, \bar{25}), (\bar{27}, \bar{6}), (\bar{27}, \bar{9}), (\bar{27}, \bar{12}), (\bar{27}, \bar{14}), \\
& (\bar{27}, \bar{15}), (\bar{27}, \bar{16}), (\bar{27}, \bar{17}), (\bar{27}, \bar{18}), (\bar{27}, \bar{19}), (\bar{27}, \bar{20}), (\bar{27}, \bar{21}), (\bar{27}, \bar{22}), \\
& (\bar{27}, \bar{23}), (\bar{27}, \bar{24}), (\bar{27}, \bar{25}), (\bar{27}, \bar{26}), (\bar{28}, \bar{10}), (\bar{28}, \bar{7}), (\bar{28}, \bar{4}), (\bar{28}, \bar{1}), \\
& (\bar{28}, \bar{2}), (\bar{28}, \bar{3}), (\bar{28}, \bar{5}), (\bar{28}, \bar{6}), (\bar{28}, \bar{8}), (\bar{28}, \bar{9}), (\bar{28}, \bar{11}), (\bar{28}, \bar{12}), \\
& (\bar{28}, \bar{13}), (\bar{28}, \bar{14}), (\bar{28}, \bar{15}), (\bar{28}, \bar{16}), (\bar{28}, \bar{17}), (\bar{28}, \bar{18}), (\bar{28}, \bar{19}), (\bar{28}, \bar{20}), \\
& (\bar{28}, \bar{21}), (\bar{28}, \bar{22}), (\bar{28}, \bar{23}), (\bar{28}, \bar{24}), (\bar{28}, \bar{25}), (\bar{28}, \bar{26}), (\bar{28}, \bar{27}),
\end{aligned}$$

The first three submatrices of  $\check{R}_q$  were given in Ref.9, but the  $56 \times 56$  submatrix is firstly obtained here.

Because we have obtained all the  $q$ -CG coefficients for the direct product of two minimal representation, it is easy to calculate the spectrum-dependent solution of the Yang-Baxter equation by the standard method <sup>4,10</sup>. From (7.61) of Ref.10 we have:

$$\begin{aligned}
\check{R}_q(x) &= (1 - xq^2)(1 - xq^{10})(1 - xq^{18}) \mathcal{P}_{2\lambda_6} + (x - q^2)(1 - xq^{10})(1 - xq^{18}) \mathcal{P}_{\lambda_5} \\
&+ (x - q^2)(x - q^{10})(x - q^{18}) \mathcal{P}_{\lambda_1} + (x - q^2)(x - q^{10})(x - q^{18}) \mathcal{P}_0
\end{aligned} \tag{4.8}$$

It coincides with the solution given in Ref.12.

By the way, in the theory of quantum groups the  $\hat{R}_q$  matrix is usually chosen as follows:

$$\hat{R}_q = q \check{R}_q^{-1} \tag{4.9}$$

where

$$\left(\check{R}_q^{-1}\right)_{bd}^{ac} = \left(\check{R}_q\right)_{db}^{ca} \Big|_{q \rightarrow q^{-1}} \quad (4.10)$$

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## References

- [1] V. G. Drinfeld, Quantum Groups, *Proceedings of the International Congress of Mathematicians*, Berkeley, 1986, Vol. 1, 798-820.
- [2] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, Quantization of Lie groups and Lie algebras, in *Algebraic Analysis*, Academic Press, 129, 1988.
- [3] C. N. Yang, *Phys. Rev. Lett.*, **19**(1967)1312; R. J. Baxter, *Ann. Phys.*, **70**(1972)193.
- [4] M. Jimbo, *Commun. Math. Phys.* **102**(1986)537.
- [5] U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, *Commun. Math. Phys.* **142**(1991)605.
- [6] P. Aschieri and Castellani, *Inter. J. Mod. Phys. A* **8**(1993)1667; Bicovariant differential geometry of the quantum group  $GL_q(3)$ , CERN-TH-6621/92; L. Castellani and M. A. R-Monteiro, A note on quantum structure constants, DFTT-18/93.
- [7] A. Kuniba, *J. Phys. A* **23**(1990)1349.
- [8] I. G. Koh and Z. Q. Ma, *Phys. Lett. B* **234**(1990)480.
- [9] J. D. Kim, I. G. Koh and Zhong-Qi Ma, *J. Math. Phys.* **32**(1991)845.
- [10] Zhong-Qi Ma, *Yang-Baxter Equation and Quantum Enveloping algebras*, Chap. 5 and Chap. 6, World Scientific, Singapore, 1993.
- [11] N. Y. Reshetikhin, Quantized universal enveloping algebras, the Tang-Baxter equation and invariants of links I, Preprint, LOMI, E-4-87, 1987.
- [12] Zhong-Qi Ma, *J. Phys. A* **23**(1990)5513.